Optimal size of a complex network

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(Received 26 September 2002; published 7 April 2003)

We investigate the response behavior of an Ising system, driven by an oscillating field, on a small-world network, with particular attention to the effects of the system size. The responses of the magnetization to the driving field are probed by means of Monte Carlo dynamic simulations with the varied rewiring probability. It is found that at low and high temperatures the occupancy ratio, measuring how many spins follow the driving field, behaves monotonically with the system size. At intermediate temperatures, on the other hand, the occupancy ratio first grows and then reduces as the size is increased, displaying a resonancelike peak at a finite value of the system size. In all cases, further increase of the size eventually leads to saturation to finite values; the size at which saturation emerges is observed to depend on the temperature, similarly to the correlation length of the system.

DOI: 10.1103/PhysRevE.67.046101 PACS number(s): 89.75.Hc, 75.10.Hk

It has been known that under appropriate circumstances, the presence of noise in a driven system can enhance rather than suppress the output of the system. Such attractive phenomena, called stochastic resonance (SR), have been widely investigated in various systems of many practical applications [1,2]. Recently, those SR phenomena have also been examined in the context of the system-size resonance: Stochastic flips of the mean field are observed to depend on the system size, leading the linear response of the system to reach a maximum at a certain system size [3]. Those works considered two systems of fully coupled noisy oscillators and one two-dimensional Ising system with nearest neighbor interactions. Namely, the underlying connection topology of dynamic variables was assumed to be regular, with either local or global connections. Meanwhile, recent studies of neuronal networks, computer networks, biochemical networks, and even social networks, have revealed that various real systems in nature possess quite complex structures, which can be described neither by regular networks nor by completely random networks [4]. Accordingly, it is desirable to study effects of the system size on collective responses in the systems with the connection topology of complex networks, which can be made more realistic. In particular, the interplay between the system size and noise may be relevant in various biological systems such as neural networks and other cell networks, which consist of finite numbers of elements. For example, in the study of stochastic resonance in biological systems, optimal sizes of calcium ion channel clusters have been examined. It is observed that the clustering of the release channels in small clusters increases the sensitivity of the calcium response [5]. This suggests a possible realization of the system, providing motivation for the investigation of the size resonance in the complex network structure.

In this paper, we consider an Ising model on a complex network, specifically, on the Watts and Strogatz (WS)-type small-world network [6]. It is well known that the WS network is characterized by a small characteristic path length $\ell \sim \ln N$, where N is the number of nodes constituting the network, and a large clustering coefficient. Both are commonly observed properties of real networks in nature. The WS network in this paper is constructed following Ref. [6]: A regular one-dimensional (1D) network of N nodes is first constructed with local connections of range k, under periodic boundary conditions. At this stage each node on the network has $2k$ nearest neighbors. Next, each local link is visited once and, with the rewiring probability P, removed and reconnected to a randomly chosen node. After a whole sweep of the entire network, the average number of shortcuts in the network of size N is given by NPk. Accordingly, the rewiring probability P may be regarded as the fraction of the average number of shortcuts over the total number of connections Nk. In this paper, the local interaction range k is set equal to two for convenience; longer ranges ($k>2$) are not expected to lead to any qualitative difference. After the WS network is built as above, an Ising spin is put on every node, and an edge (or a link) connecting two nodes is regarded as the coupling between the two spins at the two nodes. Finally, we apply an oscillatory field, driving the Ising spins, and the corresponding responses of the average spin, i.e., the magnetization are probed via Monte Carlo dynamic simulations, with attention to the effects of the system size.

The Hamiltonian for the field-driven Ising model on the WS network, which is constructed as described above, reads

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$$H = -\frac{1}{2}\sum_{i,j} J_{ij}\sigma_i\sigma_j - h(t)\sum_i \sigma_i, \tag{1}$$

where the ferromagnetic spin-spin interaction strength J_{ij} is given by

$$J_{ij} = J_{ji} \equiv \begin{cases} J & \text{for } j \in \Lambda_i \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The neighborhood Λ_i of node i stands for the set of nodes connected to i (via either local edges or shortcuts), and $\sigma_i(=\pm 1)$ represents the Ising spin at node i. The sinusoidally oscillating magnetic field $h(t) = h_0 \cos \Omega t$ is applied with the driving amplitude h_0 and frequency Ω, while the system is assumed to be in contact with an isothermal heat bath at temperature T. We probe the dynamics of the system described by Eq. (1) by means of Monte Carlo (MC) dynamic simulations, employing the heat bath single spin-flip algorithm [7] and measuring the time t in units of the MC time step. For thermalization, we start from sufficiently high temperatures and lower the temperature T slowly with the decrement $\Delta T = 0.01$ (in units of J/k_B with the Boltzmann constant k_B). The driving amplitude and frequency are taken to be $h_0 = 0.1$ and $\Omega = 0.001$. We have also considered different frequencies, for example, $\Omega = 0.01$ and 0.1, and found that the resonancelike peak indicating the system-size resonance behavior tends to diminish at higher frequencies (see below). While simulations are performed at a given temperature, the data from the first 10^5 MC steps are discarded, which turns out to be sufficient for stationarity, and measurements are made for next 10^5 MC steps. Networks of various sizes are constructed as described above, and averages are taken over 100 different network realizations for each size.

To investigate the collective response of the system, we measure the occupancy ratio R which is defined to be the average fraction of the spins in the direction of the external field [8,9]:

$$R \equiv \left\langle \frac{\text{number of spins in the direction of } h(t)}{\text{total number of spins}} \right\rangle, \tag{3}$$

where $\langle \cdots \rangle$ denotes the time average. In other words, R measures how many spins follow the oscillating magnetic field. It is easy to understand that R approaches the value 1/2 in both low- and high-temperature limits (see, e.g., Ref. [9,10]) and becomes increased near the stochastic resonance temperature, reflecting that more spins follow the external driving. Such SR phenomena have been observed in the system of given size, and it has been demonstrated that the matching condition of two time scales, the relaxation time of the system and the inverse frequency of the driving field, yields the optimal noise strength T_{SR} at which the system displays maximum responses [9,10]. Here, we consider the system studied in Ref. [10] from a different point of view and examine the behavior of the occupancy ratio with the system size at various temperatures, probing the size resonance.

In the absence of long-range shortcuts ($P=0$), the network structure reduces to that of the 1D regular network with only local couplings of range k. Accordingly, when $P=0$, the field-driven Ising model described by Eq. (1) as well as the undriven model $[h(t)=0]$ does not exhibit long-range order at finite temperatures, yielding $T_c = 0$. Note, however, that even such a 1D system displays SR behavior at finite temperatures [11]. In the presence of long-range shortcuts ($P \neq 0$), on the other hand, it has been found that both driven and undriven Ising model display ferromagnetic order at finite temperatures [10,12]; furthermore, double SR peaks, which originate from matching of two time scales have been observed.

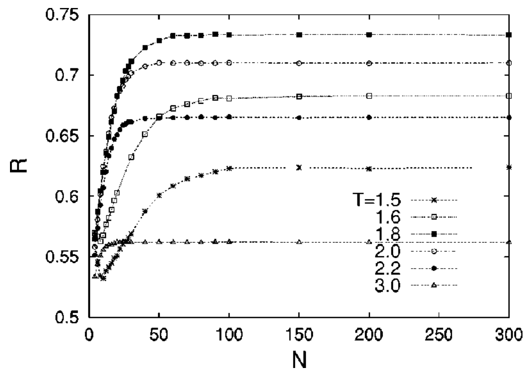

FIG. 1. Behavior of the occupancy ratio R with the system size N at various temperatures T in the absence of long-range shortcuts. Lines are merely guides to eyes.

We first consider the case without long-range shortcuts ($P=0$), i.e., the purely 1D system, and shown in Fig. 1 the behavior of the occupancy ratio R versus the system size N at various temperatures. It is observed that the occupancy ratio R first increases and eventually saturates to a finite value as the system size N is increased. The saturation size N_s, beyond which the occupancy ratio R does not increase any more, reduces as the temperature T is raised. Figure 2 displays such temperature dependence of the saturation size N_s, which has been taken as the size giving the occupancy ratio with the difference from the stationary value less than 5×10^{-4}. One can observe the exponential behavior: $N_s \propto e^{c/T}$ with $c = 5.6 \pm 0.7$, which is reminiscent of the behavior of the correlation length. In the 1D Ising model the cor-

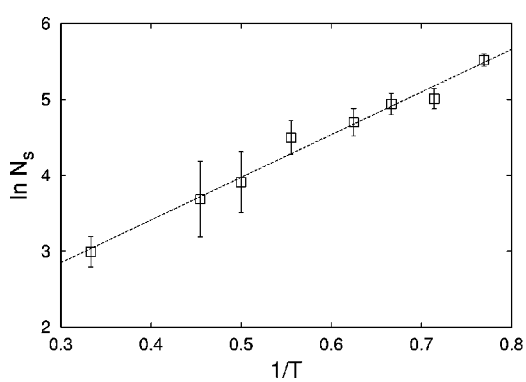

FIG. 2. Behavior of the saturation size N_s with the temperature T in the absence of shortcuts, exhibiting a linear relation between $\ln N_s$ and $1/T$, with the slope $c = 5.6 \pm 0.7$. The dotted line, obtained by the least-square fit, represents $\ln N_s = 5.6/T + 1.2$.

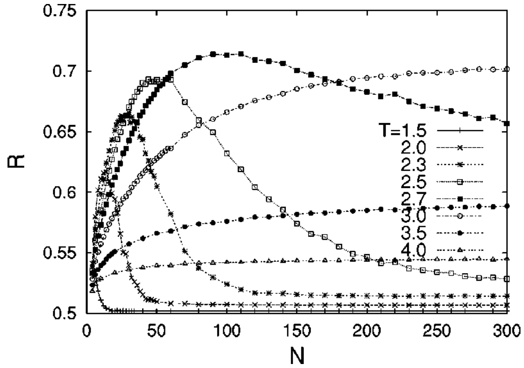

FIG. 3. The occupancy ratio R vs the system size N in the presence of the long-range shortcuts ($P=0.5$) at various temperature T. Lines are merely guides to eyes.

relation length ξ diverges in the low-temperature limit as $\xi \sim e^{k(k+1)/T}$ [12]. Here, the local interaction range k has been chosen to be two ($k=2$), leading to the behavior $\xi \sim e^{6/T}$, essentially the same as that of the saturation size. It is thus concluded that the saturation behavior emerges when the system size reaches the correlation length of the system.

Meanwhile, in the presence of long-range shortcuts ($P \neq 0$), substantially different behavior has been obtained for the occupancy ratio. In Fig. 3, the occupancy ratio R of the system with the rewiring probability $P=0.5$ is displayed as the system size N is varied. At low temperatures ($T \lesssim 1.5$), R first decreases monotonically with the size N and then saturates to the value 0.5; at high temperatures ($T \gtrsim 2.9$), on the contrary, R increases monotonically to the saturation value larger than 0.5, depending on the temperature. In contrast to these monotonic behaviors, at intermediate temperatures ($1.5 \lesssim T \lesssim 2.9$), the occupancy ratio R behaves nonmonotonically, exhibiting a maximum at a finite value of the system size N. The height of such a resonancelike peak tends to increase as the temperature is raised. We have also considered different values of the rewiring probability P as well as of the driving frequency Ω. It is found that as P is increased, the range of the temperature, in which the size-resonance behavior is displayed, becomes wider and that the saturation temperature beyond which R shows saturation behavior increases. On the other hand, as the driving frequency is increased, the position of the resonance peak shifts toward smaller values of the system size, thus tending to yield monotonic decrease of R.

Note that this resonance behavior manifests two kinds of length scale in the system with long-range shortcuts: the saturation size N_s and the resonance size N_m at which R reaches the maximum. To understand the possible relation with the correlation length even in the presence of long-range shortcuts, we examine the behaviors of N_s and N_m, which are displayed in Figs. 4–6.

Figure 4 exhibits the saturation size N_s versus the temperature T in the system with $P=0.5$. For comparison with the correlation volume described by $\xi_V \sim |T-T_c|^{-\bar{\nu}}$ [13], the data points are plotted in the logarithmic scale, thus fitted to a linear relation between $\ln N_s$ and $\ln |T-T_c|$ with the proportionality constant (slope) $\bar{\nu}$. From this fitting, we obtain $T_c \approx 2.7$, with which the slope is estimated to be $\bar{\nu}=2.0 \pm 0.1$. Noting that in d dimensions, the correlation volume relates with the correlation length ξ via $\xi_V \sim \xi^d$ and the behavior $\xi \sim |T-T_c|^{-\nu}$, we thus have $\bar{\nu}=d\nu$ in a d-dimensional system. Here, it is known that the (effective) dimension d of a mean-field system should be taken as the upper-critical dimension d_u [14], leading to $\bar{\nu}=d_u \nu$. With $d_u=4$ and $\nu=\nu_{MF}=1/2$ for a mean-field system, we conclude that the value $\bar{\nu}=2.0$ indicates a transition of the mean-field nature [13]. We also examine the other length scale N_m and show its temperature dependence in Fig. 5, where for convenience N_s, shown already in Fig. 4, is also plotted. It is shown that both the two length scales behave similarly with the temperature, yielding essentially the same value of the exponent $\bar{\nu}$

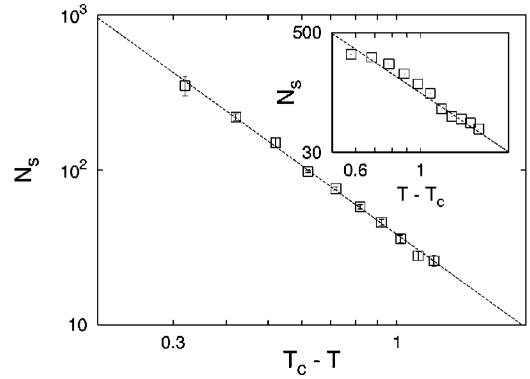

FIG. 4. Behavior of the saturation size N_s with the temperature T for $P=0.5$ at low temperatures ($T<T_c$), where the value $T_c=2.7$ obtained from the fitting has been used. The least-square fit represented by the dashed line corresponds to $\ln N_s=-2.0 \ln(T_c-T)+1.6$, the slope of which gives $\bar{\nu}=2.0$. Inset: Behavior of N_s at temperatures higher than T_c. The dashed line, obtained from the least-square fit, is given by $\ln N_s=-2.0 \ln(T-T_c)+2.1$, where the slope again leads to $\bar{\nu}=2.0$. The error bars have sizes not larger than the symbol size.

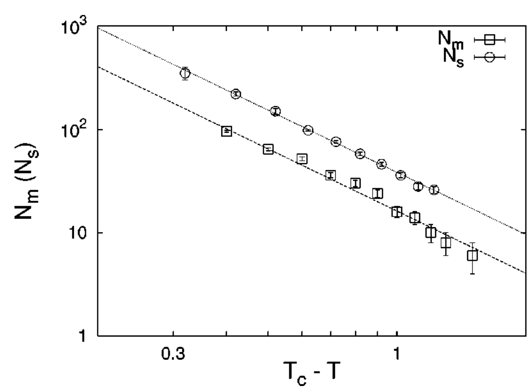

FIG. 5. The resonance size N_m together with the saturation size N_s vs the temperature T in the presence of shortcuts ($P=0.5$). The values of T_c, obtained from the fitting, are given by 3.1 and 2.7 for N_m and N_s, respectively. The dotted line represents the corresponding least-square fit of N_m, which is described by $\ln N_m=-2.0 \ln(T_c-T)+1.2$. Thus, both N_m and N_s result in the same value $\bar{\nu}=2.0$.

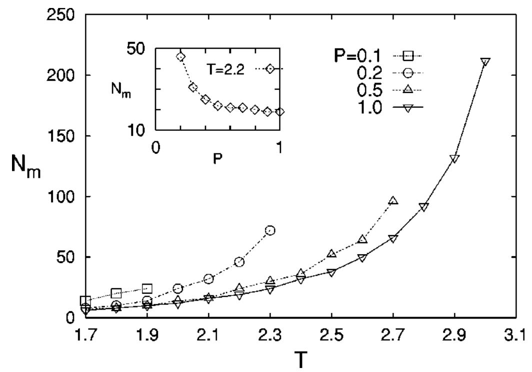

FIG. 6. The resonance size N_m vs the temperature T at various values of the rewiring probability P. Inset: Behavior of N_m with the rewiring probability P at temperature $T=2.2$.

$=2.0\pm 0.1$ (see the slopes of the two fitted lines). Accordingly, both the two length scales N_s and N_m apparently measure the correlation length of the system. On the other hand, the fitting parameter T_c for N_m turns out to be 3.1, which is somewhat higher than the value 2.7 obtained from fitting of N_s. At this stage it is difficult to discern unambiguously the difference; more extensive simulations and careful analysis are necessary for confirming the origin as well as the presence of this discrepancy.

Finally, we consider systems with different rewiring probabilities, and examine how the rewiring probability P affects the resonance behavior. Figure 6 shows the behavior of the resonance size N_m with the temperature at various values of P. It is observed that N_m first increases slowly with the temperature T then very fast as T approaches T_c, which again reminds us of the behavior of the correlation length. Behavior of N_m with the rewiring probability P at given temperature $T=2.2$ is displayed in the inset of Fig. 6. Note the rather fast decrease of N_m for small rewiring probabilities ($P \lesssim 0.5$) and the saturation behavior for large rewiring probabilities ($P \gtrsim 0.5$). Such saturation behavior has also been reported in the synchronization of the system of coupled oscillators on a WS network [15]. It is also noteworthy that N_m decreases as shortcuts are introduced to the system; the shortcuts tend to decrease the optimal system size which corresponds to the maximum collective response at given temperature and rewiring probability.

In conclusion, we have investigated the effects of the system size on the collective response, measured by the occupancy ratio, in the oscillatory field-driven Ising model on WS networks. In the purely one-dimensional system without long-range interactions, which does not undergo a phase transition at any finite temperature, the occupancy ratio has been found to display monotonic behavior, not exhibiting a resonance peak. As long-range interactions come into the system, on the other hand, system-size resonance, characterized by nonmonotonic behavior, has been observed to emerge, thus suggesting a possible relation between the size-resonance and a finite-temperature phase transition. The resonance size at which the occupancy ratio reaches the peak may be regarded as the optimal size of the network, in view of the maximum response. It is noteworthy that the optimal size as well as the saturation size displays temperature-dependent behavior, which is essentially the same as that of the correlation length of the system. This suggests the interesting possibility of estimating the correlation length from the stochastic resonance behavior at various sizes. Namely, the size resonance phenomena may be used as a tool to measure the correlation length. Note also that at given temperature both length scales, the optimal size and the saturation size, tend to decrease as the amount of long-range interactions is increased.

As a possible application of the system-size resonance, we suggest biological systems such as the assembly of β cells which reside in the islets of Langerhans in pancreas [16]. It is known that the β cells form clusters, each with a finite number of cells, rather than gathering together as one unit. Thus, speculated is the possibility that the function of the β cells may be optimized via the mechanism of the system-size resonance. In addition, the system-size resonance behavior may also be useful for understanding the formation of the public opinion [17].

H.H. thanks J. Lee for the privilege of using the computing facility Gene. B.J.K. was supported in part by the Korea Science and Engineering Foundation through Grant No. R14-2002-062-01000-0. M.Y.C. thanks the Korea Institute for Advanced Study for hospitality during his visit, where this work was performed, and acknowledges the partial support from the Swiss-Korean Outstanding Research Program.

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